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A Multi-Step Procedure for Asset Allocation in Case of Limited Resources

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Abstract: *Portfolio management is a process involving decision making in dynamic and unpredictable environment. Asset allocation plays a key role in this process, since the optimal use of the capital is a complex and resource-consuming problem. During our research in this field we have detected some problems that lead to biased results and one of them occurs in case of limited financial resources. In this paper a mathematical assessment of the dependence of the capital, used on asset prices is derived, and a multi-step procedure for asset allocation, aiming at optimization of the investor's utility in case of limited resources is described. The procedure is implemented as a module in a decision support system based on fuzzy logic. The paper contains comparison of the obtained test results with results from the classical Markowitz portfolio model. The conducted tests are on real data from the Bulgarian stock exchange.*

Keywords: Asset allocation, FSSAM, capital optimization, fuzzy systems.

1. Introduction

Recent economic practice clearly shows difficulties in the development of accurate and reliable analytical models for investment portfolio management. One of the problems in familiar optimization models in portfolio theory is the fact that the cardinality problem is often neglected [10].

Another problem arises from the inherent uncertainty and insecurity of the financial markets. The prices of assets traded on stock markets reflect a huge number of factors: political and economic governments' decisions, financial interests of companies, theories, strategies, projections, expectations and actions, psychological effects of speculators, natural disasters and many others. These factors constantly interact, because all (regulatory authorities, governments, bankers, investors and analysts) are involved in the process of decision-making environment, depending on numerous factors, including their own actions. Insecurity and uncertainty are an inescapable fact of financial markets reality. This uncertainty cannot be deeply studied and fully averted, at least because it is impossible to conduct experiments under identical conditions – the state of the global financial system is unique at each moment and different from the previous one.

Nevertheless, another problem in applying portfolio theory is the assumption that the expected returns of individual securities have normal or other probability distribution. Empirical studies prove that this assumption is not met [9]. Events on financial markets (e.g., change in asset prices) do not show random probabilistic nature. These events cannot be repeated, cannot be simulated and hence, cannot be predicted.

A powerful and thorough analysis of the situation can be found in [1].

These facts motivate the investors and financial managers to put their major efforts not so much in finding exact solutions, but rather in achieving efficient management. Actually this approach is applied not only in financial applications [8], but in many different areas (e.g., [11, 15]).

2. Cardinality problem

Let K be the available capital that an investor wishes to invest in a financial portfolio, consisting of shares of assets A_1, A_2, …, A_j, …, A_N. If the share of the asset A_j in the portfolio is denoted by x_j, then their sum should equal exactly one, as stated in the optimization models and theories, known from literature [2, 7, 9, 10, 13] and others, i.e.,

(1) $$\sum_{j=1}^{N} x_j = 1,$$

where N is the number of assets.

However, in a real world situation, this may not always be fulfilled.

Indeed, let x_j be the shares of the assets included in a financial portfolio obtained after solving the optimization problem with the constraint (1).

In this case the amount of money spent on the asset A_j equals x_j; K and the eventual number of equities purchased from the asset A_j is:

$$\nu_j = \left\lfloor \frac{x_j K}{P_j} \right\rfloor.$$

Herein the notation $\lfloor x \rfloor$ stands for the greatest integer that does not exceed the real number x.

Therefore, the amount of money spent on asset A_j is

$$\nu_j P_j = \left\lfloor \frac{x_j K}{P_j} \right\rfloor P_j.$$

The total amount of capital spent for constructing the portfolio is

$$K_{\mathrm{u}} = \sum_{j=1}^{N} \nu_j\, P_j = \sum_{j=1}^{N} \left\lfloor \frac{x_j K}{P_j} \right\rfloor P_j,$$

where N is the number of assets in the portfolio.

In a real situation this amount of spent capital (used capital) K_{u} does not always equal the initial capital K. It is very important to pay attention to the difference of the initial capital K and the spent capital K_{u}, especially in case of limited resources, because if one wants an optimal portfolio and obtains it from the optimization procedure (no matter which), the solution may not be feasible and the estimation of the solution may not be optimal. In the next section a mathematical approach is used for assessing the difference $K - K_{\mathrm{u}}$.

3. Assessment of capital in financial portfolios investments

For assessing $K - K_{\mathrm{u}}$, Fourier series of the function $\{x\}$ is used, where $\{x\}$ is the notation for the fraction part of the real number x.

By definition:

$$\{x\} = \begin{cases} x+1, & x \in (-1;0) \\ x, & x \in (0;1) \end{cases} \quad \text{for } x \notin \mathbb{Z},$$

and

$$\{x + k\} = \{x\} \text{ for } k \in \mathbb{Z}.$$

Therefore, the requirements for expressing a function as a Fourier series are satisfied and the function $\{x\}$ can be presented as a Fourier series [14].

After applying the corresponding formulae, the next Fourier coefficients are obtained:

$$a_0 = 1,$$
$$a_n = 0,$$
$$b_n = -\frac{1 + (-1)^n}{n\pi} = \begin{cases} 0, & n = 2k+1, \\ -\dfrac{1}{k\pi}, & n = 2k. \end{cases}$$

Therefore, the function $\{x\}$ can be presented as a Fourier series as follows:

$$\{x\} = \frac{1}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k}, \tag{2}$$

for $x \notin \mathbb{Z}$.

Now if (2) is applied to the function

(3) $$\lfloor x \rfloor = x - \{x\}$$
for each $x \notin \mathbb{Z}$, the following Fourier series is obtained:

(4) $$\lfloor x \rfloor = x - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k}.$$

Then, after applying (3) and (4) to the number of equities purchased from the asset A_j, which are equal to $\left\lfloor \frac{x_j K}{P_j} \right\rfloor$, the following equalities are derived:

$$K_{\mathrm{u}} = \sum_{j=1}^{N} \nu_j P_j = \sum_{j=1}^{N} \left\lfloor \frac{x_j K}{P_j} \right\rfloor P_j =$$

$$= \sum_{j=1}^{N}\left(\frac{x_j K}{P_j} - \frac{1}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2k\pi \frac{x_j . K}{P_j}}{k}\right) P_j =$$

$$= \sum_{j=1}^{N}\frac{x_j K}{P_j} P_j - \sum_{j=1}^{N}\frac{P_j}{2} + \sum_{j=1}^{N}\frac{P_j}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2k\pi \frac{x_j K}{P_j}}{k} =$$

$$= K\sum_{j=1}^{N} x_j - \frac{1}{2}\sum_{j=1}^{N} P_j + \frac{1}{\pi}\sum_{j=1}^{N} P_j \sum_{k=1}^{\infty}\frac{\sin 2k\pi \frac{x_j K}{P_j}}{k}.$$

Now the unused capital is $K - K_{\mathrm{u}}$ and for assessing it, the proportion $\frac{K-K_{\mathrm{u}}}{K}$ is calculated:

$$\frac{K - K_{\mathrm{u}}}{K} = 1 - \frac{K_{\mathrm{u}}}{K} =$$

$$= 1 - \frac{1}{K}\left(K\sum_{j=1}^{N} x_j - \frac{1}{2}\sum_{j=1}^{N} P_j + \frac{1}{\pi}\sum_{j=1}^{N} P_j \sum_{k=1}^{\infty}\frac{\sin 2k\pi \frac{x_j . K}{P_j}}{k}\right) =$$

$$= 1 - \sum_{j=1}^{N} x_j + \frac{1}{2}\sum_{j=1}^{N}\frac{P_j}{K} - \frac{1}{\pi}\sum_{j=1}^{N}\frac{P_j}{K}\sum_{k=1}^{\infty}\frac{\sin 2k\pi \frac{x_j . K}{P_j}}{k}.$$

But the constraint (1) claims that $\sum_{j=1}^{N} x_j = 1$ and hence:

(5) $$\frac{K-K_{\mathrm{u}}}{K} = \frac{1}{2}\sum_{j=1}^{N}\frac{P_j}{K} - \frac{1}{\pi}\sum_{j=1}^{N}\frac{P_j}{K}\sum_{k=1}^{\infty}\frac{\sin 2k\pi \frac{x_j K}{P_j}}{k}.$$

Then, after applying (2) for $\left\{\frac{x_j K}{P_j}\right\}$, the following equality holds:

$$\left\{\frac{x_j K}{P_j}\right\} = \frac{1}{2} - \frac{1}{\pi} W_j,$$

where $W_j = \sum_{k=1}^{\infty} \frac{\sin 2k\pi \frac{x_j.K}{P_j}}{k}$.

But $0 < \{x\} < 1$, so that

$$0 < \frac{1}{2} - \frac{1}{\pi} W_j < 1,$$

and therefore

$$-\frac{1}{2} < -\frac{1}{\pi} W_j < \frac{1}{2},$$
$$\frac{\pi}{2} > W_j > -\frac{\pi}{2}.$$

Finally, the assessment of the infinite sum W_j is

(6) $$-\frac{\pi}{2} < W_j < \frac{\pi}{2}.$$

Now let $P_* = \min_j P_j$ and $P^* = \max_j P_j$.

Then from (5) and $P_* \leq P_j \leq P^*$ it follows

(7) $$\frac{P_* N}{2K} - \frac{P^* N W_j}{\pi K} \leq \frac{K - K_u}{K} \leq \frac{P^* N}{2K} - \frac{P_* N W_j}{\pi K}.$$

Now, after applying (6) in (7), the following limitations on the proportion of the unused capital are obtained:

$$\frac{P_* N}{2K} - \frac{P^* N}{2K} \leq \frac{K - K_{\mathrm{u}}}{K} \leq \frac{P^* N}{2K} - \frac{P_* N}{2K},$$

and therefore

$$\frac{(-P^* + P_*)N}{2K} \leq \frac{K - K_{\mathrm{u}}}{K} \leq \frac{(P^* - P_*)N}{2K}.$$

But $\frac{(-P^* + P_*)N}{2.K} < 0$ and $\frac{K - K_{\mathrm{u}}}{K} > 0$, which means that the left inequality always holds and can be omitted.

Thus, the proportion $\frac{K - K_{\mathrm{u}}}{K}$ satisfies the constraints:

(8) $$0 < \frac{K - K_{\mathrm{u}}}{K} < \frac{(P^* - P_*)N}{2K}.$$

Since P_*, P^* and N are constants, then

$$\frac{K - K_{\mathrm{u}}}{K} \to 0 \quad \Leftrightarrow \quad K \to \infty,$$

which proves the following Proposition.

Proposition. In the process of portfolio management the used capital equals exactly the initial capital if and only if the initial capital tends to infinity and the number of assets in the portfolio is fixed.

But what happens if the investor does not have unlimited financial resources? In the next sections, a procedure for the opposite case (limited financial resources) is proposed.

4. Scheme of FSSAM

Fuzzy Software System for Asset Management (FSSAM) is an independent software system which implements the procedures for collection and storage of data, evaluation of assets and construction of investment portfolios.

The application software system consists of three modules (Fig. 1):

- Data Managing Module (DMM) with the following features: submits queries to the Web server of the stock exchange automatically; extracts data from the downloaded pages; writes data to the database; fills in the missing data.

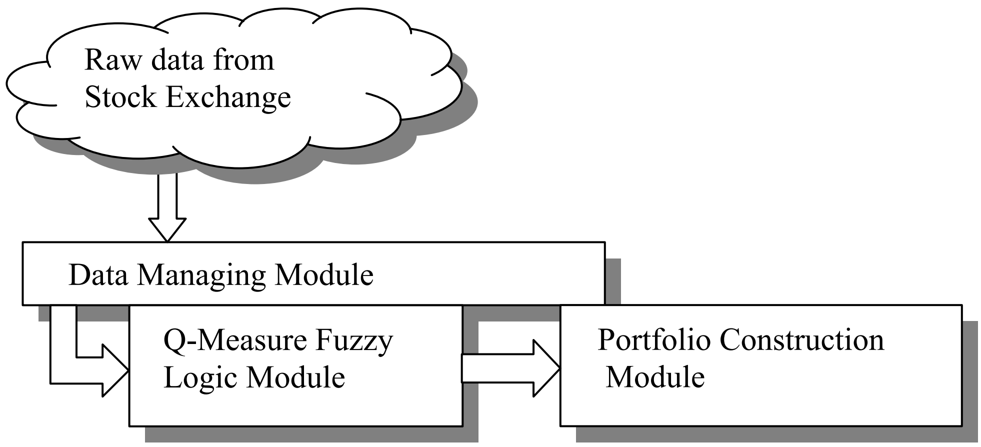

Fig. 1. Conceptual scheme of FSSAM

- Q-measure Fuzzy Logic Module (QFLM), which consists of application based on fuzzy logic. The code and the three characteristics – return, risk and q-ratio are retrieved for each asset from the database and the FLQM model is applied. FLQM model is described in full details in [6] and [12]. The input data are the crisp numerical values of the asset characteristics, obtained in DMM. These crisp values are fuzzified and after applying the aggregation rules, a fuzzy variable Q-measure for each of the assets is derived. The output is a defuzzified crisp value of Q-measure. The linguistic variables are four: three input variables and one output variable. The input variables describe the characteristics of an asset: K1 = {return}, K2 = {risk} and K3 = {q-ratio}. The output variable is Q = {Q-measure}. The input variables K1 = {return} and K2 = {risk} consist of five terms, each with the corresponding parameters: *Very low* (Sigmoid membership function), *Low* (Gaussian membership function), *Neutral* (Gaussian membership function), *High* (Gaussian membership function) and *Very high* (Sigmoid membership function). K3 consists of three terms: *Small* (Sigmoid membership function), *Neutral* (Bell membership function), and *Big* with Gaussian membership function. The output variable Q consists of five terms: *Bad*, *Not good*, *Neutral*, *Good* and *Very Good*, all with Gaussian membership functions. All fuzzy rules in this module have the form:

IF {K1 is high} AND {K2 is low} AND {K3 is big} THEN (Q is good).

There are 24 fuzzy rules implemented in the system. As a defuzzification method, the method of centre of gravity has been chosen and thus a crisp value for the asset quality is obtained as an output of QFLM.

The core part of this module is the fuzzy inference system (Fig. 2).

- A Portfolio Construction Module (PCM), which is further described.

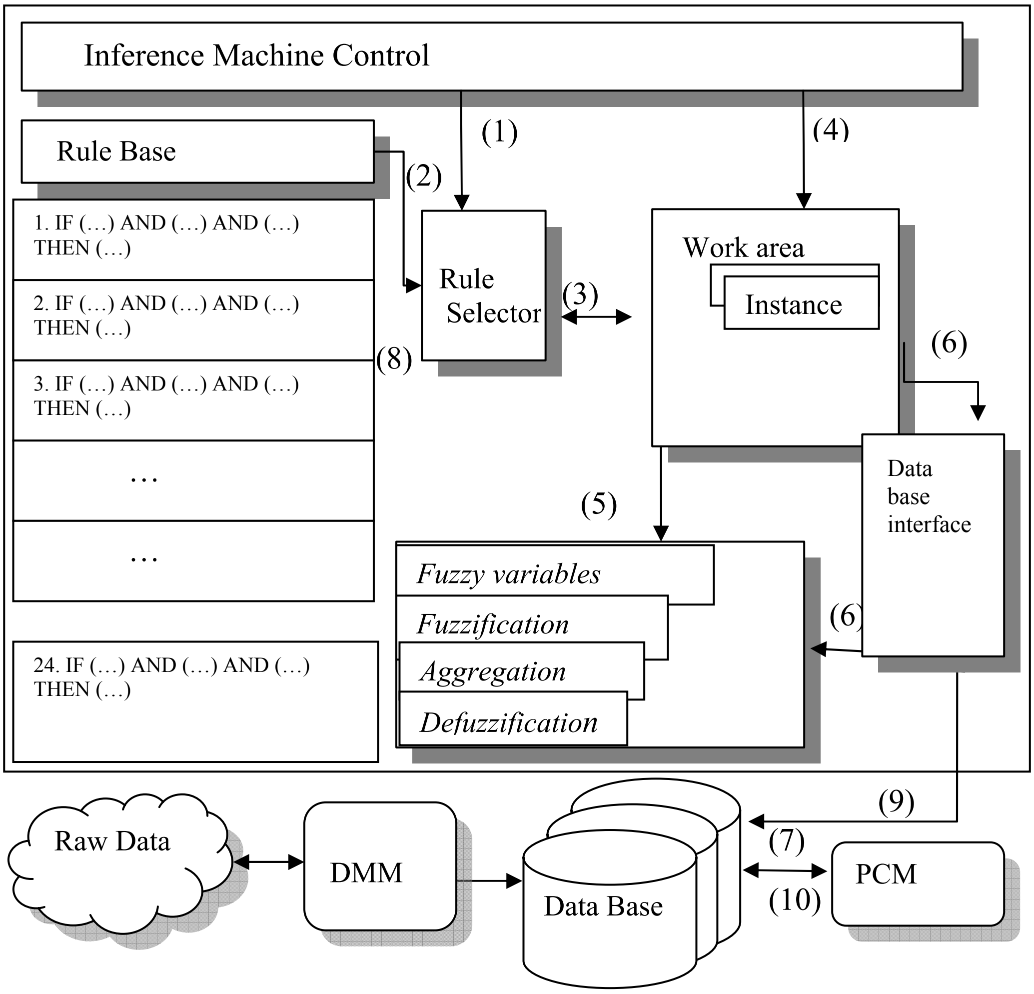

Fig. 2. Control of the inference machine, based on FLQM: selector activation (1); rule choice (2); template (3); rule activation (4); go to QFLM: Fuzzy variables, fuzzy aggregation, defuzzification (5); interface connection (6); reading from the database (7); processing the next rule (8); writing the results in the database (9); constructing a portfolio (10)

5. Multi-step optimization procedure for asset allocation

Let $A = \{A_1, A_2, A_3, \dots, A_N\}$ be a set of financial assets, K – investment capital, and M – the maximum number of financial assets to be allocated in the portfolio. The aim is to construct an investment portfolio (subset of A), which has the greatest Q and uses as much of the initial capital K as possible.

Step 1. The user enters two values in PCM: the amount of money K and the maximum number of assets M which will form the portfolios.

The code, the characteristics and the value of the output variable Q are extracted from the database.

A list of the assets sorted in a descending order according to Q is formed.

Step 2. All possible combinations without repetitions of the first m assets for any $m \leq M$ are generated: all combinations of one element from m, all combinations of two elements from m, …, all combinations of m elements from m. The number of these combinations is $2^m - 1$, since the empty set is not included. Each of the generated combinations constitutes an investment portfolio.

For each of the combinations, the asset shares x_j are calculated according to Q_j (the value of Q for the j-th asset):

$$x_j = \frac{Q_j}{\sum_{j=1}^{n} Q_j},$$

where n is the number of the assets in the portfolio.

Using these shares, the corresponding portfolios are formed and written in the database.

Step 3. An additional allocation procedure is performed if necessary and the new portfolios are added to the database.

The number v_j of the shares of A_j is calculated as

$$v_j = \left\lfloor \frac{K}{x_j P_j} \right\rfloor,$$

where x_j is obtained at Step 2, P_j is the price of A_j, and K is the investment capital.

Now the used capital K_{u} equals to:

$$K_{\mathrm{u}} = \sum_{j=1}^{n} v_j P_j.$$

In case $K - K_{\mathrm{u}}$ exceeds a preliminary set threshold, the additional allocation starts. First $K - K_{\mathrm{u}}$ is compared with the price of the asset with the highest Q and the maximum possible number of shares are bought, then this is repeated for the next asset in the list and so on, until there is no more capital left or no more assets could be bought. The number of the shares v_{j_a} is calculated as

$$v_{j_a} = \left\lfloor \frac{x_j (K - K_{\mathrm{u}})}{P_j} \right\rfloor.$$

Step 4. Three characteristics for each of the received so far portfolios are calculated:

$$R_p = \sum_{j=1}^{n} x_j r_j,$$

$$\wp_p = \sum_{j=1}^{n} x_j \sigma_j,$$

$$q_P = \frac{R_P}{\wp_P},$$

where r_j and σ_j are the return and risk of the asset A_j.

Then for each of these portfolios the QFLM is applied, using R_p as input variable K1, $\wp_p$ as K2 and q_P as K3. Thus the value of the output variable Q is obtained for all the portfolios.

Step 5. All the portfolios, portfolio characteristics, the constituent assets with the corresponding asset characteristics and shares are displayed for future use.

6. Results

In this section, as an illustration of the described model, some results obtained from real data are presented. The data are from the Bulgarian Stock exchange, so the used currency is BGN. The *Portfolio FSSAM* is the one with the highest Q amongst all the portfolios obtained from our model, and for comparison reasons *Portfolio 1* and *Portfolio 2* are used, taken from the efficient frontier obtained after applying the Markowitz model (using the algorithm, described in details in [3]). The initial capital is K= 100 000 BGN.

The portfolios are constructed on 06/20/2014 under the exact conditions of the used models. It is well known that any investor is interested mostly in the maximum return, so we will not be interested in measuring the risk. In the comparison the used criterion is not the way in which the portfolio return is calculated (geometric or arithmetic mean [4]), but the relative change of the invested amount of money. After the initial portfolios construction, the asset prices are observed, and the corresponding capital K is calculated as a sum of the asset price multiplied by its share in the portfolio. *Profit* is the difference between the initial capital (100 000BGN) and the portfolio value, given that it is sold on this date. As demonstrated in Table 1 and in Fig. 3, Portfolio FSSAM shows not only greater returns, but much more stable behaviour in the selected quarterly interval as well.

More results can be found in [5] and [6].

Table 1. Portfolio performance over a 3-month period

Portfolio	20/06/2014	02/07/2014	16/07/2014	15/08/2014	14/09/2014
Portf_1					
K	99976	100518	100704	96877	113759
Profit	**–24**	**518**	**704**	**–3123**	**13759**
Portf_2					
K	99969	100472	100575	95662	112658
Profit	**–31**	**472**	**575**	**–4338**	**12658**
Portf_FSSAM					
K	99935	100653	112636	112636	118862
Profit	**–65**	**717**	**3109**	**12700**	**18927**

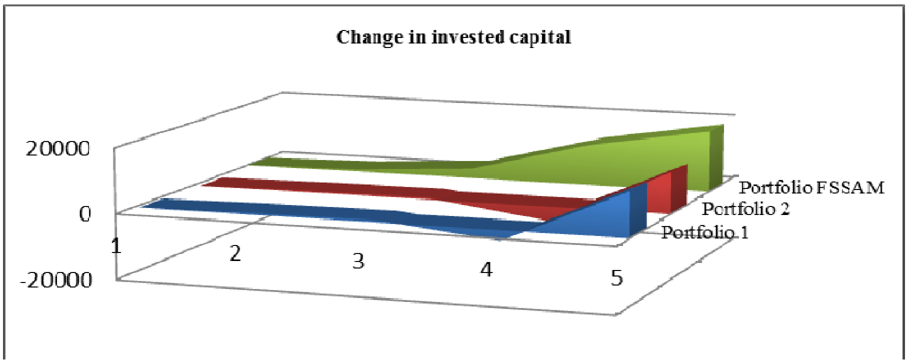

Fig. 3. Change in the invested capital (data from Table 1)

7. Conclusion

In this paper a multi-step optimization procedure for asset allocation is presented. The reason for this research and the implementation of the proposed procedure arises from a number of problems in portfolio theory applications. A strict mathematical assessment of the dependence of the capital used on asset prices; a brief description of FSSAM and comparison of the obtained test results with results from the classical Markowitz portfolio model are also described.

The proposed procedure is based on the Q-measure of an asset. The Q-measure incorporates the return, the risk and their ratio, and being modelled with fuzzy logic tools, it intuitively reflects the process of investment decisions in the economic environment with data that are often incomplete and imprecise. Although it is not based on an optimization algorithm, it solves the cardinality problem in portfolio management to a degree that suits the investors.

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References

1. B a i l e y, D. H., J. M. B o r w e i n, M. L ó p e z d e P r a d o, Q. Z h u. Pseudo-Mathematics and Financial Charlatanism: The Effects of Backtest Overfitting on Out-of-Sample Performance. – Notices of the AMS, Vol. **61**, 2014, No 5, pp. 458-471.
2. C a m p b e l l, J., A. L o, C. M a c K i n l a y. The Econometrics of Financial Markets. Princeton University Press, 1996.
3. G e o r g i e v a, P. Minimizing Risk in Investment Portfolio Management. – Biznes Posoki, Vol. **2**, 2005, pp. 35-40 (in Bulgarian).
4. G e o r g i e v a, P. Distributions of Financial Assets Characteristics. – In: Proc. of 16th International Symposium SIELA, Vol. **1**, 2009, pp. 101-106,

5. G e o r g i e v a, P., I. P o p c h e v. Cardinality Problem in Portfolio Selection. – In: M. Tomassini et al., Eds. ICANNGA'2013, LNCS 7824, Berlin, Heidelberg, Springer-Verlag, 2013, pp. 208-217
6. G e o r g i e v a, P., I. P o p c h e v. Fuzzy Q-measure Model for Managing Financial Investments.– Compt. Rend. Acad. Bulg. Sci., Vol. **66**, 2013, No 5, pp. 651-658.
7. H a u g e n, R. Modern Investment Theory. – NJ, Englewood Cliffs, Prentice-Hall, 2000.
8. I v a n o v a, Z., K. S t o i l o v, T. S t o i l o v. Portfolio Optimization – Internet Information Service. Sofia, Academic Publishing House "Prof. M. Drinov", 2005. 275 p. ISBN 954-322-021-2 (in Bulgarian).
9. J u d g e, G., R. C. H i l l, W. E. G r i f f i t h s. Introduction to the Theory and Practice of Econometrics. New York, Wiley, 1988.
10. M a r k o w i t z, H. Portfolio Selection. – The Journal of Finance, Vol. **7**, 1952, No 1, pp. 77-91.
11. P o p c h e v, I., I. R a d e v a. A Decision Support Method for Investment Preference Evaluation. – Cybernetics and Information Technologies, Vol. **6**, 2006, No 1, pp. 3-16.
12. P o p c h e v, I., P. G e o r g i e v a. A Fuzzy Approach for Solving Multicriteria Investment Problems. New York, EIAE'07, 2008, pp. 427-431.
13. S h a r p, W. Investments. NJ, Englewood Cliffs, Prentice-Hall, 1985.
14. S t e w a r t, J. Calculus. Sixth Edition. 2008. Brooks/Cole.
15. S t o y a n o v, S., I. P o p c h e v, I. D i m i t r o v. Request Globalization in an Infostation Network. – Compt. Rend. Acad. Bulg. Sci., Vol. **63**, 2010, No 6, pp. 901-908.